BAHRIA COLLEGE ZAFAR CAMPUS

E-8 ISLAMABAD

Class:XII

Unit#1

Functions and Limits

Ex#1.2

Solved Exercise 1.2

Q1 the real valued function $f(x)$ and $g(x)$ are defined below. Find (a) $fog(x)$, (b) $gof(x)$, (c) $fof(x)$, (d) $gog(x)$

I. $f(x) = 2x + 1$, $g(x) = \frac{3}{x-1}, x \neq 1$

II. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x^2}, x \neq 0$

III. $f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$ $g(x) = (x^2 + 1)^2$

IV. $f(x) = 3x^4 - 2x^2$, $g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

I. $f(x) = 2x + 1$, $g(x) = \frac{3}{x-1}, x \neq 1$

Solution

(a) $f(x) = 2x + 1$

And $g(x) = \frac{3}{x-1}, x \neq 1$

$\Rightarrow$ $fog(x) = f[g(x)] = f[\frac{3}{x-1}]$

$= 2\left[\frac{3}{x-1}\right] + 1$

$= \frac{6}{x-1} + 1$

$= \frac{x+5}{x-1}$

II. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x^2}, x \neq 0$

$f(x) = \sqrt{x+1}$

And $\quad g(x) = \frac{1}{x^2}, x \neq 0$

$fog(x) = f[g(x)]$

$= f[\frac{1}{x^2}]$

$= \frac{1}{\sqrt{\frac{1}{x^2}+1}}$

$= \sqrt{\frac{x^2+1}{x^2}}$

$= \frac{\sqrt{x^2+1}}{x}$

iii. $f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1 \qquad g(x) = (x^2+1)^2$

$f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$

And $\quad g(x) = (x^2+1)^2$

$fog(x) = f[g(x)]$

$= \frac{1}{\sqrt{(x^2+1)^2-1}}$

$= \frac{1}{\sqrt{(x^4+2x^2+1)-1}}$

$= \frac{1}{\sqrt{(x^4+2x^2}}$

$= \frac{1}{x\sqrt{x^2+2}}$

Iv. $f(x) = 3x^4 - 2x^2, \qquad g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$f(x) = 3x^4 - 2x^2$

And $\quad g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$fog(x) = f[g(x)]$

$= f[\frac{2}{\sqrt{x}}]$

$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$

$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$

$= \frac{48}{x^2} - \frac{8}{x}$

$= \frac{8(6-x)}{x^2}$

(b) The real valued function g and f are defined below. Find $gof(x)$

I. $f(x) = 2x + 1, \qquad g(x) = \frac{3}{x-1}, x \neq 1$

$f(x) = 2x + 1$

And $g(x) = \frac{3}{x-1}, x \neq 1$

$gof(x) = g[f(x)]$

$= g[2x + 1]$

$= \frac{3}{2x+1-1}$

$= \frac{3}{2x}$

II. $f(x) = \sqrt{x+1}, \qquad g(x) = \frac{1}{x^2}, x \neq 0$

$f(x) = \sqrt{x+1}$

And $g(x) = \frac{1}{x^2}, x \neq 0$

$gof(x) = g[f(x)]$

$= g[\sqrt{x+1}]$

$= \frac{1}{[\sqrt{x+1}]^2}$

$= \frac{1}{x+1}, \quad x \neq -1$

III. $f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$ $\qquad g(x) = (x^2 + 1)^2$

$f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$

And $\qquad g(x) = (x^2 + 1)^2$

$gof(x) = g[f(x)]$

$= g[\frac{1}{\sqrt{x-1}}]$

$= \left[(\sqrt{x-1})^2 + 1\right]^2$

$= \left(\frac{x}{x-1}\right)^2$

Iv. $f(x) = 3x^4 - 2x^2,$ $\qquad g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$f(x) = 3x^4 - 2x^2$

And $\qquad g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$gof(x) = g[f(x)]$

$= g\left[\frac{2}{\sqrt{3x^4 - 2x^2}}\right], \quad x \neq \sqrt{\frac{2}{3}}$

$= x\frac{2}{\sqrt{3x^2 - 2}}$

(c) The real valued function f and g are defined below. Find $fof(x)$.

I. $f(x) = 2x + 1,$ $\qquad g(x) = \frac{3}{x-1}, x \neq 1$

$$f(x) = 2x + 1$$

And $$g(x) = \frac{3}{x-1}, x \neq 1$$

$$fof(x) = f[f(x)]$$

$$= f[2x + 1]$$

$$= (2x + 1)$$

$$= 4x + 3$$

II. $f(u) = \sqrt{x + 1},$ $g(u) = \frac{1}{x^2}, x \neq 0$

$$fof(x) = f[f(x)]$$

$$f(x) = f[\sqrt{x + 1}]$$

And

$$= f[\sqrt{x + 1}]$$

$$\sqrt{\sqrt{x + 1} + 1}$$

III. $f(u) = \frac{1}{\sqrt{x-1}}, x \neq 1$ *and* $g(u) = (x^2 + 1)^2$

$$fof(u) = f[f(u)]$$

$$= f[\frac{1}{\sqrt{x-1}}]$$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}}}}$$

$$= \frac{1}{\sqrt{1-\frac{\sqrt{x-1}}{\sqrt{x-1}}}}$$

$$= \sqrt{\frac{(x-1)}{\sqrt{1-\sqrt{x-1}}}}$$

iv. $f(x) = 3x^4 - 2x^2,$ *and* $g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$$faf(n) = f[f(x)]$$

$$= f[3x^4 - 2x^2]$$

$$= 3[3x^4 - 2x^2]^4 - 2[3x^4 - 2x^2]^2$$

$$= [3x^4 - 2x^2]^2[3[3x^4 - 2x^2]^2 - 2]$$

$$= [3x^4 - 2x^2]^2[27x^8 - 36x^6 + 12x^4 - 2]$$

Or $3[3x^4 - 2x^2]^4 - 2[3x^4 - 2x^2]^2$

(d) The real valued function f and g are defined below $gog(x)$.

I. $f(x) = 2x + 1,$ $g(x) = \frac{3}{x-1}, x \neq 1$

$$f(x) = 2x + 1$$

And $g(x) = \frac{3}{x-1}, x \neq 1$

$$gog(x) = g[g(x)]$$

$$= g[\frac{3}{x-1}]$$

$$= \frac{3}{\frac{3}{x-1} - 1}$$

$$= \frac{3}{\frac{3-x+1}{x-1}}$$

$$= \frac{3(x-1)}{4-x}$$

II. $f(x) = \sqrt{x+1},$ $g(x) = \frac{1}{x^2}, x \neq 0$

$f(x) = \sqrt{x+1}$

And $g(x) = \frac{1}{x^2}, x \neq 0$

$gog(x) = g[g(x)]$

$= g[\frac{1}{x^2}]$

$= \frac{1}{[\frac{1}{x^2}]^2}$

$= \frac{1}{\frac{1}{x^4}}$

$= x^4$

III. $f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$ $g(x) = (x^2+1)^2$

$f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1$

And $g(x) = (x^2+1)^2$

$fog(x) = g[g(x)]$

$= g[(x^2+1)^2]$

$= ([(x^2+1)^2]^2 + 1)^2$

$= [(x^2+1)^4 + 1]^2$

$= (x^4 + 2x^2 + 2)^2$

Iv. $f(x) = 3x^4 - 2x^2,$ $g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$f(x) = 3x^4 - 2x^2$

And $g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

$gog(x) = g[g(x)]$

$= g\left[\frac{2}{\sqrt{x}}\right]$

$= \frac{2}{\sqrt{\frac{2}{\sqrt{x}}}}$

$= \sqrt{2\sqrt{x}}$

Q2 For the real valued function, f defined below, find

(a) $f^{-1}(x)$. (b) $f^{-1}(-1)$. and (c) $verify\ f\left[(f^{-1})(x)\right] = f^{-1}[f(x)] = x$

$f(x) = -2x + 8$ $\qquad$ $f(x) = 3x^3 + 7$

$f(x) = (-x + 9)^3$ $\qquad$ $f(x) = \frac{2x+1}{x-1}, x > 1$

Solution

I. $f(x) = -2x + 8$

Given $\quad y = f(x) = -2x + 8$

$\Rightarrow f'(y) = \frac{1}{2}(8 - y)$

i.e. $f'(x) = \frac{1}{2}(8 - x)$

s/t $f'(-1) = \frac{1}{2}(8 + 1) = \frac{9}{2}$

to verify $f = [f'(x)]$

$= f'[f(x)] = x$

$L.H.S = f[f'(x)]$

$= f\left[\frac{1}{2}(x - x)\right]$

$= \frac{-2}{2}(8 - x) + 8$

$= -8 + x + 8$

$= x = R.H.S$

Middle one $= f'[f(x)]$

$= \frac{1}{2}(2x)$

$= \frac{1}{2}[(8 - (-2x + 8)]$

$= 8 - x$

$= x = R.H.S$

II. $f(x) = 3x^3 + 7$

Given $y = f(x) = 3x^3 + 7$

$\frac{1}{3}(y - 7) = x^3$

$[\frac{1}{3}(y - 7)]^{\frac{1}{3}} = x = f'(y)$

i.e. $f'(x) = [\frac{1}{3}(y - 7)]^{\frac{1}{3}}$

. $f'(-1) = [\frac{1}{3}(-1 - 7)]^{\frac{1}{3}}$

$= \frac{1}{3}(-8) = \left[\frac{-8}{3}\right]^{\frac{1}{3}}$

To verify $f = [f'(x)]$

$= f'[f(x)] = x$

$L.H.S = f[f'(x)]$

$= [\frac{1}{3}(x - 7)]^{\frac{1}{3}}$

$= 3[(\frac{x-7}{3})^{\frac{1}{3}}]^3 + 7$

$= 3\left(\frac{x-7}{3}\right) + 7$

$= x - 7 + 7$

$= x = R.H.S$

Middle one $= f'[f(x)]$

$= f'[3x^3 + 7]$

$= [\frac{1}{3}(3x^3 + 7 - 7)]^{\frac{1}{3}}$

$= [x^3]^{\frac{1}{3}}$

$= x = R.H.S$

III. $f(x) = (-x + 9)^3$

Given $y = f(x) = (-x + 9)^3$

$y^{\frac{1}{3}} = -x + 9$

$x = 9 - y^{\frac{1}{3}}$

$f'(y) = 9 - y^{\frac{1}{3}}$

i.e. $f'(x) = 9 - x^{\frac{1}{3}}$

$f'(-1) = 9 - (-1)^{\frac{1}{3}}$

To verify $f[f'(x)]$

$= f'[f(x)] = x$

$L.H.S = f[f'(x)]$

$= f[9 - x^{\frac{1}{3}}]$

$= [9 - [9 - x^{\frac{1}{3}}]]^3$

$= [x^3]^{\frac{1}{3}}$

$= x = R.H.S$

Middle one $= f'[f(x)]$

$= f'[(9 - x)^3]^{\frac{1}{3}}$

$= 9 - (9 - x)$

$= x = R.H.S$

Iv. $f(x) = \frac{2x+1}{x-1}, x > 1$

Given $y = f(x) = \frac{2x+1}{x-1}$

$y = \frac{2x+1}{x-1}, \ x > 1$

$y(x-1) = 2x + 1$

$yx - 2x = y + 1$

$(y-2)x = y + 1$

$x = \frac{y+1}{y-2}$

$f'(y) = \frac{y+1}{y-2}$

$f'(x) = \frac{x+1}{x-2}$

$f'(-1) = \frac{-1+1}{-1-2}$

$= \frac{0}{-3}$

$= 0$

To verify $f[f'(x)] = f'[f(x)] = x$

$L.H.S = f[f'(x)]$

$= f[\frac{x+1}{x-2}]$

$= \frac{2(\frac{x+1}{x-2})+1}{(\frac{x+1}{x-2})}$

$= \frac{2(x+1)+(x-2)}{(x+1)-(x-2)}$

$= \frac{2x+2+x-2}{x+1-x+2}$

$$= \frac{3x}{3}$$

$$= x = R.H.S$$

Middle one $= f'[f(x)]$

$$= f'[\frac{2x+1}{x-1}]$$

$$= \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2}$$

$$= \frac{(2x+1)+(x-1)}{(2x+1)-2(x-1)}$$

$$= \frac{2x+1+x-1}{2x+1-2x+2}$$

$$= \frac{3x}{3}$$

$$= x = R.H.S$$

Q3 Without finding the inverse, state the domain and range of $f^{-}(x)$.

I. $f(x) = \sqrt{x+2}$

II. $f(x) = \frac{1}{x+3}, x \neq -3$

III. $f(x) = \frac{x-1}{x-4}, x \neq 4$

Iv. $f(x) = (x-5)^2, x \geq 5$

I. $f(x) = \sqrt{x+2}$

Given $f(x) = \sqrt{x+2}$

Evidently F is not defined when $x < -2$

Hence, domain F is $x \geq -2$ i.e.

$[-2, \infty]$

As a varies is over the interval $[-2, \infty]$

Hence, value of $\sqrt{x+2}$ varies over the interval $[0,\infty]$

i.e. Range $f = [0,\infty]$

now by def of inverse function f'

Range $f'(x)$ = Domain $f(x)$ = the set of real numbers except $x \geq 2$

II. $f(x) = \frac{x-1}{x-4}, x \neq 4$

Given $f(x) = \frac{x-1}{x-4}, x \neq 4$

Evidently f is undefined when $x = 4$

Hence, Domain f is $\forall X \in R$ but, $x \neq 4$

i.e. Domain $f =]-\infty, 0[\cup]0, \infty[$

and $\frac{1}{x-4}$ will vary over the interval

$]-\infty, 0[\cup]0, \infty[\Rightarrow R-\{4\}$

Which becomes the range of f

Now by def of inverse function f'

$\Rightarrow$Domain of f' = range of $f =]-\infty, 0[\cup]0, \infty[$

And range of $f' = f =]-\infty, 4[\cup]4, \infty[$

Range = Domain $f'(x)$ =set of real no. s except $x = 4$

III. $f(x) = \frac{1}{x+3}, x \neq -3$

Given $f(x) = \frac{1}{x+3}, x \neq -3$

Evidently f is undefined when $x = -3$

Hence, Domain f is $\forall X \in R$ but, $x \neq -3$

i.e. Domain $f =]-\infty, -3[\cup]-3, \infty[$

and $\frac{1}{x+3}$ will vary over the interval

$]0, -\infty[\cup]\infty, 0[$

Which becomes the range of f

Now by def of inverse function f'

$\Rightarrow$Domain of f' = range of $f =]0, -\infty[\cup]\infty, 0[$

And range of $f' = f =]-\infty, -3[\cup]-3, \infty[$

Range = Domain $f'(x)$ =set of real no. s except $x = -3$

Iv. $f(x) = (x-5)^2, x \geq 5$

Given $f(x) = (x-5)^2, x \geq 5$

Although f is defined by $\forall x \in R$

But under designed fact $x > 5 \in R$

Hence , Domain f is $\forall X \in R$ but , $x \geq 5$

i.e. Domain $f = [5, \infty[$

and x will vary over the interval $[5, \infty[$

hence value of $(x-5)^2$, varies over every positive real number and zero.

i.e. Range f is $\forall x \in R$ & $x \geq 0$

Now by def of inverse function f'

$\Rightarrow$Domain of f' = range of $f = [0, \infty[$

And range of $f' = f = [5, \infty[$

Range of $f'(x)$ = Domain $f(x)$ =set of real no. s containing $x \geq 5$