BAHRIA COLLEGE ZAFAR CAMPUS

E-8 ISLAMABAD

Class:10^{th}

Unit#2

Cube Roots of unity
And their properties

Ex#2. 2

Cube Roots of Unity

Let x be cube root of unity

Then $x^3 = 1$

$\Rightarrow \quad x^3 - 1 = 0$

$\Rightarrow \quad (x-1)(x^2+x+1) = 0$

$\Rightarrow \quad x - 1 = 0 \quad or \quad x^2 + x + 1 = 0$

$\Rightarrow \quad x = 1 \quad or \quad x^2 + x + 1 = 0$

Now solving; $x^2 + x + 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{3}\, i}{2}$$

Thus cube roots of unity are 1 , $\frac{-1+\sqrt{3}\, i}{2}$ and $\frac{-1-\sqrt{3}\, i}{2}$

Note $\quad \omega = \frac{-1+\sqrt{3}\, i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}\, i}{2}$

Properties of Cube Roots of Unity

(i) Sum of all cube roots of unity is zero.
(ii) Product of all cube roots of unity is 1.
(iii) Each complex cube root of unity is square of the other.

Proofs: (i) Sum of all cube roots of unity is zero.

Since $1, \omega \text{ and } \omega^2$ are cube roots of unity

Now; $1 + \omega + \omega^2 = 1 + \left(\frac{-1+\sqrt{3}\,i}{2}\right) + \left(\frac{-1-\sqrt{3}\,i}{2}\right)$

$$= \frac{2-1+\sqrt{3}\,i-1-\sqrt{3}\,i}{2}$$

$$= \frac{0}{2}$$

$$= 0$$

(ii) Product of all cube roots of unity is 1.

Since $1, \omega \text{ and } \omega^2$ are cube roots of unity

$$1.\omega.\omega^2 = 1.\left(\frac{-1+\sqrt{3}\,i}{2}\right).\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$1.\omega.\omega^2 = \left(\frac{-1+\sqrt{3}\,i}{2}\right).\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$1.\omega.\omega^2 = \left(\frac{1-3i^2}{4}\right)$$

$$1.\omega.\omega^2 = \left(\frac{1+3}{4}\right) \qquad \because i^2 = -1$$

$$1.\omega.\omega^2 = \frac{4}{4}$$

$1.\omega.\omega^2 = 1$ as required **<u>Note</u>:** $\omega^3 = 1$

Exercise 2.2

(i) Find the cube roots of -1,8, -27,64.

Solution:

Let $x^3 = -1$

$(x)^3 + (1)^3 = 0$

$(x+1)(x^2 - x + 1) = 0$

Either $x + 1 = 0$ *or* $x^2 - x + 1 = 0$

$x = -1$ *Here* $a = 1, b = -1, c = 1$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = \frac{-1 + \sqrt{-3}}{2} \quad or \quad x = \frac{-1 - \sqrt{-3}}{2}$$

$$= -\left(\frac{-1 - \sqrt{-3}}{2}\right) \qquad = -\left(\frac{-1 + \sqrt{-3}}{2}\right)$$

$$= -\omega^2 \qquad = -\omega$$

Three cube roots of -1 are -1, $-\omega, -\omega^2$

(ii) The three cube roots of 8

Solution:

Let $x^3 = 8$

$x^3 - 8 = 0$

$(x)^3 - (2)^3 = 0$

$(x-2)(x^2+2x+4) = 0$

Either $x - 2 = 0$ *or* $x^2 + 2x + 4 = 0$

$x = 2$ *Here* $a = 1, b = 2, c = 4$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2(-1 \pm i\sqrt{3})}{2} \because i = \sqrt{-1}$$

$$x = \frac{2(-1 + i\sqrt{3})}{2} \quad or \quad x = 2\left(-\frac{1 - i\sqrt{3}}{2}\right)$$

$$= 2\omega \qquad = 2\omega^2$$

Three cube roots of 8 are $2, 2\omega, 2\omega^2$

(iii) The three cube roots of -27

Solution:

Let $x^3 = -27$

$x^3 + 27 = 0$

$(x)^3 - (3)^3 = 0$

$(x+3)(x^2 - 3x + 9) = 0$

Either $x + 3 = 0$ *or* $x^2 - 3x + 9 = 0$

$x = -3$ *Here* $a = 1, b = -3, c = 9$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$x = \frac{3(-1 \pm i\sqrt{3})}{2} \because i = \sqrt{-1}$$

$$x = \frac{3(1 + i\sqrt{3})}{2} \quad or \quad x = 3\left(-\frac{1 - i\sqrt{3}}{2}\right)$$

$$x = \frac{-3(-1 - i\sqrt{3})}{2} \quad or \quad x = -3\left(\frac{-1 + i\sqrt{3}}{2}\right)$$

$= -3\omega^2$ $= -3\omega$

Three cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) The three cube roots of 64

Solution:

Let $x^3 = 64$

$x^3 - 64 = 0$

$(x)^3 - (4)^3 = 0$

$(x-4)(x^2+4x+16) = 0$

Either $x - 4 = 0$ *or* $x^2 + 4x + 16 = 0$

$x = 4$ *Here* $a = 1, b = 4, c = 16$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16-64}}{2}$$

$$x = \frac{-4 \pm \sqrt{-48}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{-3}}{2}$$

$$x = \frac{4\left(-1 \pm i\sqrt{3}\right)}{2} \because i = \sqrt{-1}$$

$$x = \frac{3\left(-1 + i\sqrt{3}\right)}{2} \quad or \quad x = 4\left(\frac{-1 - i\sqrt{3}}{2}\right)$$

$$= 4\omega \qquad\qquad = 4\omega^2$$

Three cube roots of 64 are $4, 4\omega, 4\omega^2$

2. Evaluate

(i) $\left(1-\omega-\omega^2\right)^7$

Solution:

$$\left(1-\omega-\omega^2\right)^7=\left[1-\left(\omega+\omega^2\right)\right]^7$$
$$=\left[1-1(-1)\right]^7 \quad \because \omega+\omega^2=-1$$
$$=(1+1)^7$$
$$=2^7=128$$

(ii) $\left(1-3\omega-3\omega^2\right)^5$

Solution:

$$\left(1-3\omega-3\omega^2\right)^5=\left[1-3\left(\omega+\omega^2\right)\right]^5$$
$$=\left[1-3(-1)\right]^5 \because \omega+\omega^2=-1$$
$$=(1+3)^5$$
$$=4^5=1024$$

(iii) $\left(9+4\omega+4\omega^2\right)^3$

Solution:

$$\left(9+4\omega+4\omega^2\right)^3=\left[9+4\left(\omega+\omega^2\right)\right]^3$$
$$=\left[9+4(-1)\right]^3 \because \omega+\omega^2=-1$$
$$=(9-4)^3$$
$$=5^3=125$$

(vi) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$

Solution:

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$$

$=\omega^9+\left(2\omega^2\right)^9 \qquad \because \omega=\frac{-1+\sqrt{-3}}{2}$ *and* $\omega^2=\frac{-1-1\sqrt{-3}}{2}$

$=\omega^9+\omega^{18}$

$=\left(\omega^3\right)^3+\omega^{18}$

$=\left(\omega^3\right)^3+\left(\omega^3\right)^6=(1)^3+(1)^6 \qquad \because \omega^3=1$

$=1+1=2$

(vii) $\omega^{37}+\omega^{38}-5$

Solution:

$\omega^{37}+\omega^{38}-5$

$=\omega^{37}+\omega^{38}-5$

$=\omega^{36}.\omega+\omega^{36}.\omega^2-5$

$=\left(\omega^3\right)^{12}.\omega+\left(\omega^3\right)^{12}.\omega^2-5$

$=(1)^{12}.\omega+(1)^{12}.\omega^2-5 \qquad \because \omega^3=1$

$=\omega+\omega^2-5$

$=-1-5 \qquad \because \omega+\omega^2=-1$

$=-6$

(viii) $\omega^{-13}+\omega^{-17}$

Solution:

$\omega^{-13}+\omega^{-17}$

$=\omega^{-13}+\omega^{-17}$

$=\omega^{-12-1}+\omega^{-15-2}$

$=\omega^{12}.\omega^{-1}+\omega^{-15}.\omega^{-2}$

$=\left(\omega^{3}\right)^{-4}.\omega^{-1}+\left(\omega^{3}\right)^{-5}.\omega^{-2}$

$=(1)^{-4}.\omega^{-1}+(1)^{-5}.\omega^{-2}$ $\quad\because \omega^{3}=1$

$=\omega^{-1}+\omega^{-2}$

$=\dfrac{1}{\omega}+\dfrac{1}{\omega^{2}}$

$=\dfrac{\omega^{2}+\omega}{\omega^{3}}$ $\quad\because \omega+\omega^{2}=-1\ and\ \omega^{3}=1$

$=\dfrac{-1}{1}$

$=-1$

3. Prove that $x^{3}+y^{3}=(x+y)(x+\omega y)\left(x+\omega^{2}y\right)$.

Solution:

$$x^3 + y^3 = (x+y)(x+\omega y)(x+\omega^2 y)$$

$$\begin{aligned}
R.H.S &= (x+y)(x+\omega y)(x+\omega^2 y) \\
&= (x+y)\left[x(x+\omega^2 y)+\omega y(\omega+\omega^2 y)\right] \\
&= (x+y)\left[x^2+\omega^2 xy+\omega xy+\omega^3 y^2\right] \\
&= (x+y)\left[x^2+(\omega^2+\omega)xy+(1)y^2\right] \quad \because \omega^3 = 1 \\
&= (x+y)\left[x^2+(-1)xy+y^2\right] \quad \because \omega^2+\omega=-1 \\
&= (x+y)(x^2-xy+y^2) \\
&= L.H.S
\end{aligned}$$

Hence Proved.

4. Prove that $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.

Solution:

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$$

$$\begin{aligned}
R.H.S &= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \\
&= (x+y+z)\left[x(x+\omega^2 y+\omega z)+\omega y(x+\omega^2 y+\omega z)+\omega^2 z(x+\omega^2 y+\omega z)\right] \\
&= (x+y+z)\left[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+\omega^2 xz+\omega^4 yz+\omega^3 z^2\right] \\
&= (x+y+z)\left[x^2+\omega^2 xy+\omega xy+\omega^2 yz+\omega yz+\omega^2 xz+\omega xz+(1)y^2+(1)z^2\right] \\
&= (x+y+z)\left[x^2+(\omega^2+\omega)xy+(\omega^2+\omega)yz+(\omega^2+\omega)xz+y^2+z^2\right] \quad \because \omega^2 = 1 \\
&= (x+y+z)\left[x^2+(-1)xy+(-1)yz+(-1)xz+y^2+z^2\right] \quad \because \omega^2+\omega=-1 \\
&= (x+y+z)\left[x^2+y^2+z^2-xy-yz-zx\right] \\
&= x^3+y^3+z^3-3xz \\
&= L.H.S
\end{aligned}$$

Hence Proved.

5. Prove that $(i+\omega)(i+\omega^2)(i+\omega^4)(i+\omega^8)$.....2n factors=1.

Solution:

$$L.H.S = (i+\omega)(i+\omega^2)(i+\omega^4)(i+\omega^8).....\text{2n factors}$$

$$= (i+\omega)(i+\omega^2)(i+\omega^3.\omega)(i+\omega^2.\omega^6).....\text{2n factors}$$

$$= (i+\omega)(i+\omega^2)(i+\omega^3.\omega)\left(i+\omega^2.(\omega^3)^2\right).....\text{2n factors}$$

$$= (i+\omega)(i+\omega^2)(i+(1)\omega)\left(i+\omega^2(1)^2\right).....\text{2n factors} \because \omega^3 = 1$$

$$= (i+\omega)(i+\omega^2)(i+\omega)(i+\omega^2).....\text{2n factors}$$

$$= (-\omega^2)(i+\omega^2)(-\omega^2)(i+\omega^2).....\text{2n factors} \qquad \because 1+\omega = -\omega^2$$

$$= \left[(-\omega^2)(-\omega)\right]\left[(-\omega^2)(-\omega)\right].....\text{n factors} \qquad \because 1+\omega^2 = -\omega$$

$$= \left[\omega^3\right]\left[\omega^3\right].....\text{n factors}$$

$$= (1)(1).....\text{n factors} \qquad \because \omega^3 = 1$$

$$= (1)^n$$

$$= 1$$

$$= R.H.S$$

Hence Proved.