Limits of important functions.

1.5.1 $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

where n is an integer and $a > 0$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

$$L.H.S = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

Put $x = a + h$

as $x \to a$, $h \to 0$

$$= \lim_{h \to 0} \frac{(a+h)^n - a^n}{a + h - a}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[(a+h)^n - a^n \right]$$

$\because (a+b)^n = a^n + n a^{n-1} \cdot b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + b^n$

$$L.H.S = \lim_{h\to 0} \frac{1}{h}\left[a^n + n a^{n-1}.h + \frac{n(n-1)}{2!} a^{n-2}.h^2 + \cdots + h^n - a^n \right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[n.a^{n-1}.h + \frac{n(n-1)}{2!} a^{n-2}h^2 + \cdots + h^n \right]$$

$$= \lim_{h\to 0} \frac{h}{h}\left[n a^{n-1} + \frac{n(n-1)}{2!} a^{n-2}.h + \cdots + h^{n-1} \right]$$

$$= n a^{n-1} + \lim_{h\to 0} a^{n-2}.h + \cdots + \lim_{h\to 0} h^{n-1}$$

$$= n a^{n-1}$$

$$= R.H.S$$

nce proved

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

1.5.5 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

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$L.H.S = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$\therefore (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots+$

$L.H.S = \lim_{n\to\infty}\left[1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\cdots\right]$

$= \lim_{n\to\infty}\left[1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\cdots\right]$

$= \lim_{n\to\infty}\left[2+\frac{1}{2!}\left(\frac{n-1}{n}\right)+\frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)+\cdots\right]$

$= \lim_{n\to\infty}\left[2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\right]$

$$L.H.S = \lim_{n\to\infty}\left[2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{2}{n}\right)\cdot\left(1-\frac{1}{n}\right) + \cdots\right]$$

when $n \to \infty$, $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$ ... all tends to zero

$$L.H.S = \lim\left[2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right]$$

$$= 2 + 0.5 + 0.1667 + \cdots$$

$$= 2.71201$$

$$= e$$

$$= R.H.S$$

Hence proved

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

1.5.6 $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

L.H.S $= \lim_{x \to 0} \frac{a^x - 1}{x}$

Put $a^x - 1 = y$

$a^x = 1 + y$

Taking log on both sides

$\log(a^x) = \log(1+y)$

$x \log(a) = \log(y+1)$

$x = \frac{\log(y+1)}{\log(a)}$

$$\boxed{x = \log_a(y+1)}$$

as $x \to 0$, $y \to 0$

L.H.S $= \lim_{y \to 0} \frac{y}{\log_a(y+1)}$

$= \lim_{y \to 0} \frac{1}{\frac{1}{y}\log_a(1+y)}$

$$L.H.S = \lim_{y \to 0} \frac{1}{\log_a (1+y)^{1/y}}$$

$$= \frac{1}{\lim_{y \to 0} \log_a (1+y)^{1/y}}$$

$$= \frac{1}{\log_a \lim_{y \to 0} (1+y)^{1/y}}$$

$$= \frac{1}{\log_a (e)} \qquad \because \lim_{x \to 0} (1+x)^{1/x} = e$$

$$= \log_e a$$

$$= R.H.S$$

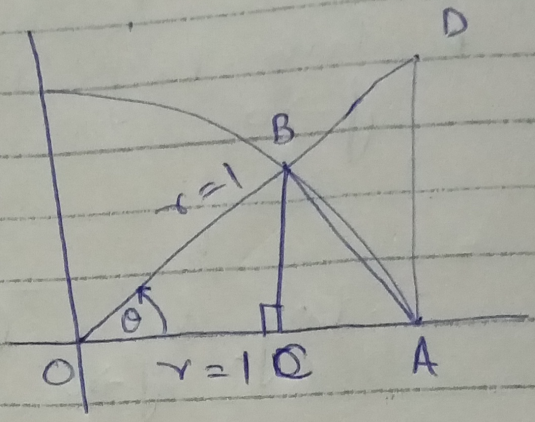
hence proved.

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$$

1.5.8 If θ is measured in Radian then $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Proof:

To evaluate this limit, we apply technique. Take θ a positive acute central angle of a circle with radius $r = 1$.



Area of $\Delta OAB <$ Area of sector $OAB <$ Area of ΔOAD. $\rightarrow$ (A)

(i) Area of $\Delta OAB = \frac{1}{2}$ (base)(height)

$= \frac{1}{2}(|OA|)(|BC|)$

$= \frac{1}{2}|BC|$

$\because |OA| = 1$

consider ΔOBC

$\sin\theta = \frac{|\bar{BC}|}{|\bar{OB}|}$

$= \frac{|\bar{BC}|}{1}$ $\because |\bar{OB}| = 1$

$$\boxed{|\bar{BC}| = \sin\theta}$$

$$\rightarrow \boxed{\text{Area of } \Delta OAB = \frac{1}{2}(\sin\theta)}$$

$$\text{Area of sector } OAB = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2}(1)(\theta) \quad \because r = 1$$

$$\rightarrow \boxed{\text{Area of sector } OAB = \frac{1}{2}\theta}$$

$$\text{Area of } \Delta OAD = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(|\bar{OA}|)(|\bar{AD}|)$$

$$= \frac{1}{2}(1)(|\bar{AD}|)$$

$$= \frac{1}{2}(\tan\theta)$$

Put the required values in eq (A)

Area of $\Delta OAB <$ Area of sector $OAB <$ Area of ΔOAD

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\sin\theta < \theta < \tan\theta$$

÷ by $\sin\theta$

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

when $\theta \to 0$

$$\lim_{\theta\to 0}(1) > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > \lim_{\theta\to 0}(\cos\theta)$$

$$1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > 1$$

By using sandwich theorem

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$$