BAHRIA COLLEGE ZAFAR CAMPUS

E-8 ISLAMABAD

Class:XII

Unit#1

Functions and Limits

Ex#1.3

Theorems on limits:

1. $\lim_{x \to a}[g(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) + g(x)]$
3. $\lim_{x \to a}[Kf(x)]$
4. $\lim_{x \to a}[f(x) + g(x)]$
5. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
6. $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$

Theorem: prove that $\lim_{x \to \infty}(1 + \frac{1}{n})^n = e$

Proof by Binomial Theorem

$$= (1 + \frac{1}{n})^n = 1 + n\left(\frac{1}{n}\right)\frac{n(n-1)}{2}(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{3}(\frac{1}{n})^3$$

$$= \left(1 + 1\frac{1}{2}\right)n^2\left(n - \frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \frac{n^3\left(n - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{1}\frac{1}{n^3}$$

Now A,n→ ∞, $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$ all to zero

$\Rightarrow \quad \lim_{x \to \infty}(1 + \frac{1}{n})^n = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots\ldots$

$$= 1 + 1 + 0.5 + 0.166667 + 0.416667 + \cdots$$

$$\leq 2.718287$$

$$\lim_{x \to \infty}(1 + \frac{1}{n})^n = e$$

Solved Exercise 1.3

Q1 Evaluate each limit by using theorem of limits.

Solution

i. $\lim_{x\to 3}(2x+4)$

$= \lim_{x\to 3}(2x) + \lim_{x\to 3} 4$

$= 2\lim_{x\to 3}(x) + \lim_{x\to 3} 4$

$= 2(3) + 4$

$= 6 + 4 = 10$

ii. $\lim_{x\to 1}(3x^2 - 2x + 4)$

$= \lim_{x\to 1}(3x^2) + \lim_{x\to 1}(2x) + \lim_{x\to 1}(4)$

$= 3\lim_{x\to 1}(x^2) + 2\lim_{x\to 1}(x) + \lim_{x\to 1}(4)$

$= 3(1)^2 - 2 + 4$

$= 3 - 2 + 4$

$= 5$

iii. $\lim_{x\to 3}\sqrt{x^2 + x + 4}$

$= \lim_{x\to 3}(x^2) + \lim_{x\to 3}(x) + \lim_{x\to 3}(4)$

$= (3)^2 + 3 + 4$

$= (16)^{\frac{1}{2}}$

$= 4$

iv. $\lim_{x \to 2} x\sqrt{x^2 - 4}$

$= (\lim_{x \to 2} x)(\lim_{x \to 2} \sqrt{x^2 - 4})$

$= (2)(\sqrt{4 - 4})$

$= (2)(\sqrt{0})$

$= 0$

v. $\lim_{x \to 2}(\sqrt{x^3 + 1} - \sqrt{x^2 + 5})$

$= \lim_{x \to 2}(x^3 + 1)^{\frac{1}{2}} - \lim_{x \to 2}(x^2 + 5)^{\frac{1}{2}}$

$= (\sqrt{2^3}) + 1 - (\sqrt{2^2}) + 5$

$= \sqrt{9} - \sqrt{9} = 0$

vi. $\lim_{x \to -2} \frac{2x^3+5x}{3x-2}$

$= \frac{\lim_{x \to \ 2} 2x^3+5x}{\lim_{x \to \ 2} 3x-2} = \frac{2\lim_{x \to \ 2} x^3+5 \lim_{x \to \ 2} x}{3 \lim_{x \to \ 2} x - \lim_{x \to \ 2} 2}$

$= \frac{2(-2)^3+(5(-2))}{3(-2)-2}$

$= \frac{-16-10}{-6-2}$

$= \frac{-26}{-8} = \frac{13}{4}$

Q2 Evaluate each limit by using algebraic techniques.

i. $\lim_{x \to -1} \frac{x^3-x}{x+1}$

$= \lim_{x \to -1} \frac{x(x^3-x)}{x+1}$

$= \lim_{x \to -1} \frac{x(x^2-1)}{x+1}$

$= \lim_{x \to -1} \frac{x(x-1)(x+1)}{x+1}$

$= \lim_{x \to -1} x(x-1)$

$= (-1)(-1-1)$

$= 2$

ii. $\lim_{x \to 1} \frac{3x^3+4x}{x^2+x}$

$= \frac{3(1)^3+4(1)}{(1)^2+(1)}$

$= \frac{3+4}{1+1}$

$= \frac{7}{2}$ (Book answer is 4)

iii. $\lim_{x \to 2} \frac{x^3-8}{x^2-x-6}$

$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x^2-3x+2x-6}$

$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x(x-3)+2(x-3)}$

$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-3)}$

$= \lim_{x \to 2} \frac{(2-2)(2^2+(2)(2)+4)}{(2+2)(2-3)}$

$= 0$

iv. $\lim_{x\to 1}\frac{x^3-3x^2+3x-1}{x^3-x}$

$$=\lim_{x\to 1}\frac{(x^3-1)-3x^2+3x}{x(x^2-1)}$$

$$=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)-3x(x-1)}{x(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{(x-1)(x^2+x+1-3x)}{x(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{(x^2-2x+1)}{x(x+1)}$$

$$=\lim_{x\to 1}\frac{(x-1)^2}{x(x+1)}$$

$$=\frac{(1-1)^2}{1(1+1)}=\frac{0}{2}=0$$

v. $\lim_{x\to -1}(\frac{x^3+x^2}{x^2-1})$

$$=\lim_{x\to -1}(\frac{x^2(x+1)}{(x+1)(x-1)})$$

$$=\lim_{x\to -1}(\frac{x^2}{(x-1)})$$

$$=(\frac{(-1)^2}{(-1-1)})$$

$$=\frac{(-1)^2}{(-2)}$$

$$=\frac{-1}{2}$$

vi. $\lim_{x\to 4}\frac{2x^2-32}{x^3-4x^2}$

$$=\lim_{x\to 4}\frac{2(x^2-16)}{x^2(x-4)}$$

$$=\lim_{x\to 4}\frac{2(x^2-16)}{x^2(x-4)}$$

$= \lim_{x \to 4} \frac{2(x-4)(x+4)}{x^2(x-4)}$

$= \lim_{x \to 4} \frac{2(x+4)}{x^2}$

$= \frac{2(4+4)}{4*4}$

$= \frac{16}{16} = 1$

vii. $\lim_{x \to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$

$= \lim_{x \to 2} \frac{\sqrt{x}-\sqrt{2}}{(\sqrt{x})^2-(\sqrt{2})^2}$

$= \lim_{x \to 2} \frac{1}{\sqrt{x}+\sqrt{2}}$

$= \frac{1}{\sqrt{2}+\sqrt{2}}$

$= \frac{1}{2\sqrt{2}}$

viii. $\lim_{x \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{x \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} * \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{x \to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{x \to 0} \frac{1}{(\sqrt{x+h})} + \sqrt{x}$

$= \frac{1}{(\sqrt{0+h})} + \sqrt{0}$

$= \frac{1}{(\sqrt{h})}$

ix. $\lim_{x \to a} \frac{x^n-a^n}{x^m-a^m}$

Let $x = a + h$

$x = a$

$a = a + h$

$h = 0$

$$= \lim_{x \to a} \frac{(a+h)^n - a^n}{(a+b)^m - a^m}$$

$$= \lim_{x \to a} \frac{a^n \left[\left(1 + \frac{h}{a}\right)^n - 1\right]}{a^m \left[\left(1 + \frac{b}{a}\right)^m - 1\right]}$$

Using binomial series

$$= \lim_{h \to 0} \frac{a^n \left[1 + \frac{n}{1}\left(\frac{b}{a}\right)^1 + \frac{n(n\ \ 1)}{2}\left(\frac{b}{a}\right)^c + -1\right]}{a^m \left[1 + \frac{m}{1}\left(\frac{b}{a}\right)^1 + \frac{m(m\ \ 1)}{2}\left(\frac{b}{a}\right)^c + -1\right]}$$

$$= \lim_{h \to 0} \frac{a^n \left[n\left(\frac{n}{a}\right) + \frac{(n\ \ 1)b^2}{2\ \ a^2} + m\right]}{a^m \left[\left(\frac{n}{a}\right) + \frac{m(m\ \ 1)b^2}{2\ \ a^2} + m\right]}$$

Taking h common

$$= \lim_{h \to 0} \frac{a^n \left[\left(\frac{n}{a}\right) + \frac{n(n\ \ 1)\ b}{2\ \ ac} + \right]}{a^m \left[\left(\frac{m}{a}\right) + \frac{m(m\ \ 1)\ b}{2\ \ a^2} + \right]}$$

$$= \frac{a^n \left[\left(\frac{n}{a}\right) + 0\right]}{a^m \left[\left(\frac{m}{a}\right) + 0\right]}$$

$$= \frac{a^n . \frac{n}{a}}{a^m . \frac{m}{a}}$$

$$= \frac{n}{m} . a^{n-m}$$

Q3 Evaluate the following limits.

I. $\lim\limits_{h\to 0}\frac{sin7x}{x}$

$=\lim\limits_{h\to 0}\frac{sin7x}{\frac{7x}{7}}$

$=7\lim\limits_{h\to 0}\frac{sin7x}{7x}$

$=7(1)$

$=7$

II. $\lim\limits_{x\to 0}\frac{sinx^0}{x}$

As $180^0=\pi rad$

$1^0=\frac{\pi}{180^0}rad$

$x^0=x\frac{\pi}{180^0}$ put in above

$=\lim\limits_{x\to 0}\frac{sinx\frac{\pi}{180^0}}{x}.\frac{\frac{\pi}{180^0}}{\frac{\pi}{180^0}}$

$=\lim\limits_{x\to 0}\frac{sin\frac{x\pi}{180^0}}{\frac{x\pi}{180^0}}.\frac{\pi}{180^0}$

$=(1).\frac{\pi}{180^0}$

$=\frac{\pi}{180^0}$

III. $\lim\limits_{x\to 0}\frac{1-cos\theta}{sin\theta}$

$=\lim\limits_{x\to 0}\frac{1-cos\theta}{sin\theta}.\frac{1+cos\theta}{1+cos\theta}$

$=\lim\limits_{x\to 0}\frac{1-cos^2\theta}{sin\theta(1+cos\theta)}$

$$= \lim_{x \to 0} \frac{sin^2\theta}{sin\theta(1+cos\theta)}$$

$$= \lim_{x \to 0} \frac{sin\theta}{(1+cos\theta)}$$

$$= \frac{0}{1+1} = \frac{0}{2} = 0$$

iv. $\lim_{x \to \pi} \frac{sin\theta}{\pi - x}$

Let $x = \pi - \theta$

As $x \to \pi, \theta \to 0$

$$= \lim_{\theta \to 0} \frac{\sin(\pi) - \theta}{\pi - (\pi - \theta)}$$

But $\sin(\pi - \theta)$

$$= sin\theta$$

$$= \lim_{\theta \to 0} \frac{sin\theta}{\theta}$$

$$= 1$$

v. $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

$$= \lim_{x \to 0} \left[\left(\frac{bx}{\sin bx}\right)\left(\frac{\sin ax}{ax}\right)\left(\frac{ax}{bx}\right)\right]$$

$$= \lim_{x \to 0} \left[\left(\frac{\sin ax}{ax}\right)\right] . \lim_{x \to 0} \left[\left(\frac{1}{\frac{\sin bx}{bx}}\right)\left(\frac{a}{b}\right)\right]$$

$$= \left(\frac{a}{b}\right)(1) - \frac{1}{1}$$

$$= \left(\frac{a}{b}\right)$$

vi. $\lim_{x \to 0} \frac{x}{\tan x}$

$= \lim_{x \to 0} \frac{\cos x.x}{\sin x}$

$= \lim_{x \to 0} \frac{\cos x}{\frac{\sin x}{x}} = \lim_{x \to 0} \frac{\cos x.x}{\sin x}$

$= (1)\cos 0^0$

$= (1)(1)$

$= 1$

vii. $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

$= \lim_{x \to 0} \frac{1-(2\cos^2 x - 1)}{x^2}$, $\quad \cos 2x = 2\cos^2 x - 1$

$= \lim_{x \to 0} \frac{(2\cos^2 x)}{x^2}$

$= \lim_{x \to 0} \frac{(\sin^2 x)}{x^2}$

$= 2\lim_{x \to 0} \frac{\sin x}{x}$

$= 2(1)$

$= 2$

viii. $\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$

$= \lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x} \cdot \frac{1 + \cos x}{1 + \cos x}$

$= \lim_{x \to 0} \frac{\sin^2 x}{\sin^2 x(1 + \cos x)}$

$= \lim_{x \to 0} \frac{1 + \cos x}{1 + \cos x}$

$= \lim_{x \to 0} \frac{1}{1 + \cos x}$

$= \frac{1}{1 + \cos 0}$

$$= \frac{1}{1+1} = \frac{1}{2}$$

ix. $\lim\limits_{\theta \to 0} \frac{sin^2\theta}{\theta}$

$$= \lim_{\theta \to 0} \frac{sin^2\theta}{\theta} . \theta$$

$$= \lim_{\theta \to 0} \frac{sin^2\theta}{\theta} . \lim_{\theta \to 0}(0)$$

$$= (1^2)(0) = 0$$

x. $\lim\limits_{x \to 0} \frac{sec\, x - cos\, x}{x}$

$$= \lim_{x \to 0} \frac{sec\, x - cos\, x}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1 - cos^2 x}{cos\, x}}{x}$$

$$= \lim_{x \to 0} \frac{1 - cos^2 x}{x.cos\, x}$$

$$= \lim_{x \to 0} \frac{sin^2 x}{x.cos\, x}$$

$$= \lim_{x \to 0} \frac{sin\, x}{x} . tan\, x$$

$$= \lim_{x \to 0} \frac{sin\, x}{x} \lim_{x \to 0} tan\, x$$

$$= (1)(0) = 0$$

xi. $\lim\limits_{\theta \to 0} \frac{1 - cosp\theta}{1 - cosq\theta}$

$$= \lim_{\theta \to 0} \frac{2sin^2\frac{p\theta}{2}}{2sin^2\frac{q\theta}{2}}$$

$$= \lim_{\theta \to 0} sin\ \frac{2pq}{2} . \frac{1}{sin\theta} . (\frac{q\theta}{2})^2$$

$$= \lim_{\theta \to 0} (\frac{sin\frac{p\theta}{2}}{\frac{p\theta}{2}}) \lim_{\theta \to 0} (\frac{\frac{1}{sin\frac{q\theta}{2}}}{\frac{q\theta}{2}}) \frac{\frac{p^2}{4}}{\frac{q^2}{4}}$$

$= 1.1\frac{p^2}{q^2}$

$= \frac{p^2}{q^2}$

xii. $\lim_{\theta \to 0} \frac{tan\theta - sin\theta}{sin^3\theta}$

$= \lim_{\theta \to 0} \frac{sin\theta - sin\theta cos\theta}{\frac{cos\theta}{sin^3\theta}}$

$= \lim_{\theta \to 0} \frac{sin\theta(\frac{1 \;\; cos\theta}{cos\theta})}{sin^3\theta}$

$= \lim_{\theta \to 0} \frac{1 - cos\theta}{sin^2\theta - cos\theta}$

$= \lim_{\theta \to 0} \frac{1 - cos\theta}{(1+cos\theta)(1-cos\theta).cos\theta}$

$= \lim_{\theta \to 0} \frac{cos\theta}{(1+cos\theta)}$

$= \frac{cos\theta}{(1+cos\theta)}$

$= \frac{cos0.cos0}{(1+cos\theta)}$

$= \frac{1}{1+1}.1$

$= \frac{1}{2}$

Q4 Express each limit in terms of e.

i. $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{2n}$

$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{2n}$

$= \lim_{n \to \infty} (\left(1 + \frac{1}{n}\right)^{n})^2$ $\qquad \left(1 + \frac{1}{n}\right)^{n} = e$

$= e^2$

ii. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$

$= \lim_{n\to\infty}(\left(1+\frac{1}{n}\right)^{n})^{\frac{1}{2}}$ $\left(1+\frac{1}{n}\right)^{n} = e$

$= e^{\frac{1}{2}}$

iii. $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$

$= \lim_{n\to\infty}\left(1+(-\frac{1}{n})\right)^{-(-n)}$

$= \lim_{n\to\infty}\left(1+(-\frac{1}{n})^{-n}\right)^{(-1)}$

$= \lim_{n\to\infty} e^{(-1)}$

$= e^{(-1)}$

iv. $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{n}$

$= \lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{3.\frac{n}{3}}$

$= \lim_{n\to\infty}\left((1+\frac{1}{3n})^{3n}\right)^{\frac{1}{3}}$

$= e^{\frac{1}{3}}$

v. $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{n}$

$= \lim_{n\to\infty}\left(\left(1+\frac{4}{n}\right)^{\frac{n}{4}}\right)^{4}$

$= e^4$

vi. $\lim_{x \to 0}(1 + 3x)^{\frac{2}{x}}$

$= \lim_{x \to 0}(1 + 3x)^{\frac{2}{3x} \cdot 3}$

$= \lim_{x \to 0}((1 + 3x)^{\frac{1}{3x}})^6$

$= e^6$

vii. $\lim_{x \to 0}(1 + 2x^2)^{\frac{1}{x^2}}$

$= \lim_{x \to 0}((1 + 2x^2)^{\frac{1}{2x^2}})^2$

$= e^2$

viii. $\lim_{h \to 0}(1 - 2h)^{\frac{1}{h}}$

$= \lim_{h \to 0}(1 + (-2h))^{\frac{1}{2h} \cdot (-2)}$

$= e^{-2}$

ix. $\lim_{x \to 0}(\frac{x}{1+x})^x$

$= \lim_{x \to 0}(\frac{1+x}{x})^{-x}$

$= \lim_{x \to 0}(1 + \frac{1+x}{x})^{-x}$

$= \lim_{x \to 0}((1 + \frac{1}{x})^x)^{-1}$

x. $\lim\limits_{x \to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x < 0$

Let $y = \frac{1}{x}$

As $x \to 0, y \to \infty$

$= \lim\limits_{x \to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$= \lim\limits_{y \to 0} \frac{e^{y}-1}{e^{y}+1}$

$= \frac{0-1}{0+1}$

$= -1$

xi. $\lim\limits_{x \to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x > 0$

Let $y = \frac{1}{x}$

As $x \to 0, y \to \infty$

$= \lim\limits_{x \to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$= \lim\limits_{x \to 0} \frac{1-\frac{1}{e}y}{1+\frac{1}{e}y}$

$= \frac{1-0}{1+0}$

$= 1$