Ex #2.4

Q#1: If α, β are the roots of the equation $x^2 + px + q = 0$, then evaluate

(i) $\alpha^2 + \beta^2$

As α, β be the roots of the equation $x^2 + px + q = 0$

$$\alpha + \beta = -\frac{b}{a}$$

$$\boxed{\alpha + \beta = -p}$$

$$\alpha\beta = \frac{c}{a}$$

$$\boxed{\alpha\beta = q}$$

(i) $\alpha^2 + \beta^2$

$$\alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

Put $\alpha + \beta = -p$ & $\alpha\beta = q$

$$\alpha^2 + \beta^2 = (-p)^2 - 2(q)$$

$$\boxed{\alpha^2 + \beta^2 = p^2 - 2q}$$

(ii) $\alpha^3\beta + \alpha\beta^3$

Solution

$$\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$$

$$= \alpha\beta(\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta)$$

$$= \alpha\beta[(\alpha+\beta)^2 - 2\alpha\beta]$$

$$= \alpha\beta[(-p)^2 - 2q]$$

$$\boxed{\alpha^3\beta + \alpha\beta^3 = q(p^2 - 2q)}$$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

Solution:

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{\alpha\beta}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-p)^2 - 2q}{q}$$

$$\boxed{\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{p^2 - 2q}{q}}$$

Q#2: If α, β are the roots of the equation $4x^2 - 5x + 6 = 0$, then find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

Let α, β be the roots of the equation $4x^2 - 5x + 6 = 0$

then

$$\alpha + \beta = -\frac{b}{a}$$

$$\boxed{\alpha + \beta = \frac{5}{4}}$$

$$\alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{6}{4}$$

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$= \frac{5/\cancel{4}}{6/\cancel{4}}$$

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} = \frac{5}{6}}$$

(ii) $\alpha^2\beta^2$

$$\alpha^2\beta^2 = (\alpha\beta)^2$$

$$= \left({}^{3}\cancel{6}/\cancel{4}_{2}\right)^2$$

$$\boxed{\alpha^2\beta^2 = \frac{9}{4}}$$

(iii) $\dfrac{1}{\alpha^2\beta} + \dfrac{1}{\alpha\beta^2}$

$$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{1}{\alpha\beta}\left[\frac{1}{\alpha} + \frac{1}{\beta}\right]$$

$$= \frac{1}{\alpha\beta}\left[\frac{\alpha+\beta}{\alpha\beta}\right]$$

$$= \left(\frac{1}{6/4}\right)\left[\frac{5/\cancel{4}}{6/\cancel{4}}\right]$$

$$= \frac{\cancel{4}\,2}{\cancel{6}\,3}\left(\frac{5}{6}\right)$$

$$= \frac{\cancel{10}\,5}{\cancel{18}\,9}$$

$$\boxed{\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{5}{9}}$$

(iv) $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$

$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{\alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta) - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{1}{\alpha\beta}\left[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\right]$$

$$= \frac{1}{6/4}\left[\left(\frac{5}{6}\right)^3-3\left(\frac{\cancel{6}}{4}\right)\left(\frac{5}{\cancel{2}}\right)\right]$$

$$\boxed{\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = -\frac{235}{96}}$$

Q#3: If α, β be the roots of the equation $lx^2+mx+n=0$ then find

(i) $\alpha^3\beta^2+\alpha^2\beta^3$ (ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solution:

If α, β be the roots of the equation $lx^2+mx+n=0$

$\alpha+\beta = -\frac{b}{a}$ | $\alpha\beta = \frac{c}{a}$

$\alpha+\beta = -\frac{m}{l}$ | $\alpha\beta = \frac{n}{l}$

(i) $\alpha^3\beta^2 + \alpha^2\beta^3$

Solution:

$$\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha+\beta)$$

$$= (\alpha\beta)^2(\alpha+\beta)$$

Put $\alpha\beta = \frac{n}{l}$ & $\alpha+\beta = -\frac{m}{l}$

$$\alpha^3\beta^2 + \alpha^2\beta^3 = \left(\frac{n}{l}\right)^2\left(-\frac{m}{l}\right)$$

$$= -\frac{mn^2}{l^3}$$

(ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$$

$$= \frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

Put $\alpha + \beta = \frac{-m}{l}$ & $\alpha\beta = \frac{n}{l}$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2}\left[(\alpha+\beta)^2 - 2\alpha\beta\right]$$

$$= \left(\frac{l}{n}\right)^2 \left[\left(-\frac{m}{l}\right)^2 - 2\left(\frac{n}{l}\right)\right]$$

$$= \frac{l^2}{n^2}\left[\frac{m^2}{l^2} - \frac{2n}{l}\right]$$

$$= \frac{\cancel{l^2}}{n^2}\left[\frac{m^2 - 2nl}{\cancel{l^2}}\right]$$

$$\boxed{\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{m^2 - 2nl}{n^2}}$$